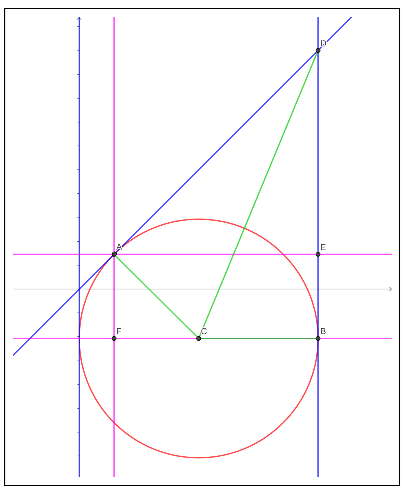
Name:

Problem 6. Find the equation of a circle which is tangent to the lines $x = 0$, $x = 2$, and $y = x$.

Solution. We begin by labeling our geometric objects as follows. We label the center (h, k). Clearly, $h = 1$, and the radius of the circle is $r = 1$. We need to find k.

- Let $L_1 = \text{locus}(x = 0)$, $L_2 = \text{locus}(x = 2)$, and $L_3 = \text{locus}(y = x)$.
- Let A be the point of tangency between the circle and L_3. Label $A = (a, a)$.
- Let B be the point of tangency between the circle and L_2, so that $B = (2, k)$.
- Let C be the center of the circle, so that $C = (h, k)$.
- Let D be the intersection of L_2 and L_3, so that $D = (2, 2)$.
- Let E be the intersection of the line $\overleftrightarrow{BD}$ and the line through A perpendicular to $\overleftrightarrow{BD}$.
- Let F be the intersection of the line $\overleftrightarrow{BC}$ and the line through A perpendicular to $\overleftrightarrow{BC}$.

Please refer to the following diagram.



Since the slope of L_3 is 1, we see that the triangles $\triangle AED$ and $\triangle AFC$ are isosceles right triangles.

Since $\triangle AFC$ is a right triangle, $|AF|^2 + |CF|^2 = |AC|^2$. We see that $|AF| = |CF| = a - k$, so

$$2(a - k)^2 = 1.$$

From this, $\sqrt{2}a = 1 + \sqrt{2}k$.

Since $\triangle CBD$ is similar to $\triangle CAD$, we have $|AD| = |BD|$. Since $\triangle AED$ is a right triangle, $|AE|^2 + |DE|^2 = |BD|^2$. We see that $|AE| = |DE| = 2 - a$ and $|BD| = 2 - k$, so

$$2(2 - a)^2 = (2 - k)^2.$$

Thus $2\sqrt{2} - \sqrt{2}a = 2 - k$.

Substitution now gives $2\sqrt{2} - 1 - \sqrt{2}k = 2 - k$. Solving this for k yields

$$k = 1 - \sqrt{2}.$$

Thus the equation of the circle is

$$(x - 1)^2 + (y - (1 - \sqrt{2}))^2 = 1.$$

□